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DISCUSSION.

NON-EUCLIDEAN GEOMETRY AND THE KANTIAN *A PRIORI.*

From the days of Pythagoras and Plato down to those of Kant and Herbart the mathematical sciences, and especially geometry, have played so important a part in the discussions of philosophers as models of method and patterns of certitude, that philosophy cannot but be extremely sensitive to any change or progress occurring in the views of mathematicians. Accordingly the philosophic world was considerably startled, not so many years ago, to hear that certain mathematicians and physicists had had the audacity to question the assumptions concerning the nature of Space, which had been consecrated by the tradition of 2000 years and set forth in the geometry of Euclid. The possibilities of non-Euclidean spaces, which were as yet necessarily ill-defined and ill-understood, promptly attracted the adherents of all views for which orthodox science appeared to have no room, and no notion seemed too fantastic to become credible, if not intelligible, in space of four or more dimensions. The mathematicians themselves, who were engaged in elaborating the new conceptions, were too busy or too uncertain of their ground to resist successfully this inundation of crankiness, and the consequent discredit into which the subject fell seems to have killed the general interest in it everywhere but in France. Meanwhile mathematicians proceeded quietly with the work of analysing the new conceptions and of deducing their consequences, and thereby reached a clearer consciousness of their import. The result has been that saner views have begun to prevail, and that the sensational features of the new geometry have been mitigated or eliminated. The question has become arguable without the opposing champions considering each other respectively unintelligible cranks or unimaginative stick-in-the-muds. Not but what the rhapsodical view still periodically finds expression in print,[1] but the tendency of the interesting exchange of opinions which has been going on for the last few years in the French philosophical and scientific journals between MM. Delbœuf, Renouvier, Poincaré, Calinon, Lechalas, de Broglie, etc., seems to me to be decidedly in the direction of agreement based upon a retreat from extreme and extravagant positions on either side. In other words, the blare of trumpets which announced and advertised the arrival of the new claimant to scientific recognition is over, the pachydermatous ears of the established conservatism have recovered from the shock, and preparations are being made to assign to the newcomer a definite place in the array of the sciences.

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[1] *E.g.*, *Monist*, IV, p. 483.

The time then seems to be becoming opportune for attempting to summarize the results of all this controversy, with a view to (*a*) bringing out the most important points established by the new 'metageometry,' (*b*) considering what light they throw on the nature of Space, (*c*) estimating what changes will have to be made in the references to geometry which philosophers have been so addicted to making. It is indeed possible that the attempt is still premature, that the parties are still too bitter to be completely reconciled, that the subject is still too inchoate and chaotic for its full significance to be determined. In that case the present writer would console himself with the reflection that his efforts can at least do no harm. His arbitration is wholly unauthorized, and compromises neither the metageometers nor their opponents. His opinions, although they will be found to have most affinity with the views of Professor Delbœuf, cannot claim to rest upon the doctrine of any great authority on either side, — they are merely the subjective inferences of a spectator with divided sympathies; and if he succeeds in arousing interest and elucidatory comment, his object will have been as fully attained as he is entitled to expect.

I. I shall begin, therefore, by stating a point which the metageometers have not to my mind satisfactorily established, and that is the value of the conception of a *fourth* dimension. I say advisedly "of the conception," for the actual existence, or even the possibility of imagining a fourth dimension seems to have been practically given up. The chief value of the conception seems nowadays to be situated in the possibility of making symmetrical solids coincide by revolving them in a fourth dimension. But this seems a somewhat slender basis on which to found the conception of a fourth dimension, and the same end could apparently[1] also be achieved by means of the conception of a 'spherical' space. Here then, probably, is the reason why of late the fourth dimension has not been so prominent in the forefront of the battle, and why its place has, with a great advance in intelligibility, been taken by spherical and pseudo-spherical *three-dimensional* 'space.'

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[1] Cp. Delbœuf, *Rev. Phil.*, XIX, 4 (abstracted in PHILOSOPHICAL REVIEW, III, p. 502).

It is on rendering these latter thinkable that the non-Euclideans have concentrated efforts, and, so far as I can judge, they have, in a large measure, been successful. It has been shown that Euclidean geometry may, nay, logically must, be regarded as a special case of general geometry, and as logically on a par with spherical and pseudo-spherical geometry. It is a species of a genus, and the differentia which constitutes it is the famous "postulate of Euclid," which Euclid postulated because he could not prove it, and which the failures of all his successors have only brought into clearer light as an indispensable presupposition. The non-Euclideans, on the other hand, have shown that it does not require proof, because it embodies the definition of the sort of space dealt with by ordinary geometry; and that in both of its equivalent forms, whether as the axiom of parallels or of the equality of the angles of a triangle to two right angles, it forms a special case intermediate between that of spherical and that of pseudo-spherical space. In spherical space nothing analogous to the Euclidean parallels is to be found; in pseudo-spherical space, on the other hand, not one, but *two* 'parallels' may be drawn through any point. So while spherical triangles always have their angles *greater* than two right angles, the pseudo-spherical triangles always have them *less* than two right angles. Moreover, the Euclidean case can always be reached by supposing the 'parameter' of the non-Euclidean spaces infinitely large. So much for the possibility of a general geometry, including the Euclidean amongst others.

It has also, I think, been shown that the non-Euclidean geometries would form coherent and consistent systems, like the Euclidean, in which an indefinite number of propositions might be shown to follow from their initial definitions. They are, that is to say, thoroughly *thinkable* and free from contradiction, and intellectually on a level with the Euclidean conception of space. They are thinkable, — but (as yet) no more; and this explains their defence against the two objections upon which their more unprejudiced opponents incline to lay most stress. It is objected (1) that there is, *e.g.*, no such thing as a spherical space, only a spherical surface. True; but there is nothing to prevent us from *conceiving* the peculiar properties of a spherical surface as pervading every portion of the space it bounds. We can conceive a spherical surface of a constant curvature making up the texture of space, just as well as the Euclidean plane surface. In the latter case the homogeneity of Space is entire in *all* respects, in the former only in *some*. It is argued (2) that metageometry is dependent on Euclidean geometry, because it is reached only through the latter. But it is not clear that it may not be logically independent, even though historically it has developed out of Euclidean geometry, and even though psychologically the latter affords the simplest means of representing spatial images.

Theoretically, then, metageometry seems to be able to give a very good account of itself. But it must be confessed that this at present only accentuates its practical failure. It is admitted that Euclidean geometry yields the simplest formulas for calculating spatial relations, and even M. Calinon[1] hardly ventures to hope that non-Euclidean formulas will be found serviceable. Metageometers mostly confine themselves to supposing imaginary worlds, of which the laws would naturally suggest a non-Euclidean formulation.[2] In short, practically the supremacy of the old geometry remains incontestable, because of its greater simplicity and consequent facility of application.

II. I pass on to the second question, the light thrown by non-Euclidean geometry on the nature of Space. In this respect incomparably its most important achievement seems to have been to force upon all the distinction between *perceptual* and *conceptual* space, or rather spaces. On this point both parties are at one, and we find, *e.g.*, M. Delbœuf[3] and M. Poincaré[4] stating the characteristics of Euclidean space and its fundamental distinction from perceptual space in almost identical terms. The former is one, empty, homogeneous, continuous, infinite, infinitely divisible, identical, invariable; the latter is many, filled, heterogeneous, continuous only for perception (if the atomic view of matter holds), probably finite, not infinitely divisible and variable. Both sides agree that our physical world is neither in Euclidean nor in non-Euclidean space, both of which are conceptual abstractions; their dispute is merely as to which furnishes the proper method for calculating spatial phenomena.[5] Thus all geometrical spaces are grounded on the same experience of physical space, which they interpret differently, while seeking to simplify and systematize it by means of the various postulates which define them.

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[1] *Rev. Phil.*, XVIII, 12 (abstracted in PHILOSOPHICAL REVIEW, III, p. 369).

[2] *E.g.*, M. Poincaré, *Rev. de Mét.*, III, 6, pp. 641 ff.

[3] *Rev. Phil.*, XVIII, 11 (abstracted in PHILOSOPHICAL REVIEW, III, p. 233).

[4] *Rev. de Mét.*, III, p. 632.

[5] Cp. Calinon, *Rev. Phil.*, XVIII, 12, "Sur l'indétermination géometrique de l'univers."

But if conceptual and perceptual space are so different, have they anything in common but the name? If the former are abstracted from the latter, upon what principles and by what methods does the abstraction proceed?

I conceive the answer to this important question to be, by the same methods as those by which 'real' or physical space is developed out of the *psychological* spaces. For, as M. Poincaré[1] well shows, we form our notion of real space by fusing together the data derived from visual, tactile, and motor sensations. That fusion is largely accomplished by ignoring the differences between their several deliverances and by correcting the appearances to one sense by another, in such a manner as to give the most complete and trustworthy perception of the object. We manipulate the data of the senses in order to perceive *things* (in 'real' space), and at a higher stage the same purposive process yields conceptual space, of course at first in its simplest form, the Euclidean. And (though I have not found this stated) all the characteristics of Euclidean space may be shown to have been constructed in this manner. Just as, *e.g.*, the varying appearances of things to the different senses were ignored in order to arrive at their 'real' place, so the varying and irregular deformations to which they are subjected at different places, when abstracted from, lead to the homogeneity of space. They are slight enough to be neglected, but if they were larger and followed some definite and simple law, they might suggest a non-Euclidean geometry. Similarly, geometrical space is one and infinite, because as soon as we abolish any boundary *in thought*, we can abolish all; it is infinitely divisible, because as soon as the division is conceived of as proceeding *in thought* the same act may be repeated as often as we please. And so on; geometrical space appears throughout as a construction of the intellect, which proceeds by the ordinary methods of that intellect in the achievement of its peculiar purposes. Nor is there anything new or mysterious about the process; no new faculty need be invoked, no new laws of mental operation need be formulated.

III. That the philosophic importance of this result is capital, is surely evident. The *certainty* of geometry is thereby shown to be nothing but the certainty with which conclusions follow from non-contradictory premisses, and in each geometry it flows from the definitions. The certainty with which the sum of the angles of a triangle may be asserted to equal two right angles in Euclidean geometry, is precisely the same as that with which it may be shown to be greater or less in non-Euclidean systems.

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[1] *Loc. cit.*

And this shows that certainty in the sense of intrinsic consistency has nothing to do with the question of the real validity of a geometry. The latter depends on the possibility of systematizing our spatial experience by means of the geometry. Our experience being what it is, we find the Euclidean the simplest and most effective system, alike to cover the facts and to calculate the divergences between the ideal and the actual results; and so we use it. But if our experience were different, a non-Euclidean system might conceivably seem preferable. In short, *as applied*, a geometry is not certain, but useful.

Again, the *necessity* of geometry is simply the necessity of a logical inference, — hypothetical, and in no wise peculiar to geometry. Similarly, the *universality* of geometrical judgments is by no means peculiar to them, but may be explained as arising out of the *methodological* character of the assumptions on which they rest. If we decide to make certain assumptions because they are the most serviceable, we can certainly know beforehand that we shall always and under all circumstances judge accordingly. To expect us to do otherwise, would be to expect us to stultify ourselves. And certainly we have a great interest in upholding the universal validity of geometrical judgments. Is it a small thing to be able to draw a figure on paper in one's study, and on the strength of it, and by virtue of the homogeneity of space, to draw inferences about what happens beyond the path of the outmost sun? Should we not be incredible idiots, if we allowed any cheat of appearances to cajole us into a moment's doubt of so precious an organon of knowledge? It would seem, then, that the chief result of metageometry is to raise into clearer consciousness the nature of the complex processes whereby we organize our experiences, and to assimilate the case of space to our procedure elsewhere.

But it has already become abundantly evident that a view of Space, such as that propounded, provokes conflicts with ancient and venerable views that have long adorned the histories of Philosophy. Among them Kant's conception of the apriority of Space is preeminent.

At a cursory glance it might indeed seem as though the new geometry afforded a welcome support to the Kantian position. If Euclidean geometry alone could prove the possibility of synthetic judgements *a priori*, could enrich us with absolutely certain knowledg absolutely independent of experience, could sustain an all-embracing, though empty, form of pure intuition, surely now that it is reinforced by two or more sister sciences, a boundless extension of our *a priori* knowledge might reasonably be anticipated. Unfortunately it proves a case of 'too many cooks' and the embarrassment of riches, rather than of 'the more the merrier.' To suppose *three a priori* forms of intuition corresponding to the three geometries is evidently not feasible, for they are in hopeless conflict with each other. If it is a universal and necessary truth that the angles of a triangle are equal to two right angles, it cannot be an equally universal and necessary truth that they are greater, according as we happen to be speaking of a Euclidean or of a spherical triangle. Clearly, there must be something seriously wrong about the assumed relation of geometry to space, or about the import of the criterion of apriority. Just as the *de facto* existence of geometry seemed to Kant to prove the possibility of an *a priori* intuition of Space, so the *de facto* existence of metageometry indicates the derivative nature of an intuition Kant had considered ultimate.

And the analysis thus necessitated rapidly discovers the seat of the error. Kant, like all philosophers before and far too many since his time, regards the conception of Space as simple and primary and the word as unambiguous. He does not distinguish between physical and geometrical space, between the problems of pure and of applied geometry. Hence he is forced to make his *Anschauung* an unintelligible hybrid between a percept and a concept, to argue alternately that 'space' could not be either, and to infer that it must therefore be some third thing. The possibility that it might be both never struck him. Still less did he suspect that each of these alternatives was complex, and that perceptual space was constructed out of no less than three sensory spaces, while it was susceptible of three different conceptual interpretations. What Kant calls 'space' therefore is not really one, but *seven*, and the force of his argument is made by their union. Confined to any one of them, the argument falls to pieces. When we see these facts as clearly as the development of metageometry has compelled us to see them, we must surely confess that the Kantian account of Space is hopelessly and demonstrably antiquated and can lend no support to the rest of his system. And should we not henceforth take care to eschew the vice of talking vaguely of 'space' without specifying what kind of space we mean, whether conceptual or perceptual, and what form of each? Even pedagogically, one would think, there can no longer be any advantage in confusing what is capable of being so clearly distinguished.

It would exceed my limits if I were to try to investigate whether Kant has not been guilty of a parallel confusion between felt succession and conceptual time in his account of the latter, still more were I to discuss whether after the withdrawal of the 'forms of pure intuition' any meaning could continue to be assigned to the Kantian conception of the *a priori.* I shall conclude, therefore, with the hope that some of the many professed believers in the *Transcendental Aesthetic* will not disdain to define their position in face of the development of modern metageometry.

F. C. S. SCHILLER.